Optimum Fully Cavitated Hydrofoils at Zero Cavitation Number

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Optimum fully cavitated hydrofoils having analytic profiles, which have either minimum drag for a given lift or maximum lift for a given drag, are investigated. The case of zero cavitation number is considered using the conformal mapping procedure of Levi-Civita, a method ideally suited for the optimization process. The results indicate that profiles with maximum lift for a given drag do not exist, whereas profiles with minimum drag for a given lift exist, but are not unique. The latter optimal profiles have zero drag. It is found further that these profiles have wetted pressures that are not everywhere greater than the cavity pressure, or, in this case, greater than the freestream pressure. If the latter condition is imposed as a constraint, it is found that optimum profiles cease to exist. The optimum profiles as found, on the other hand, have a significance as ventilated profiles where one has realistic cavitation pressures (much less than the freestream pressures), but where the cavity pressure has been increased artificially to the freestream value by mass addition. Three examples have been calculated, and the resulting profile shape, the wetted pressure distribution, and the cavity streamlines are given. Lastly, a modification of the Levi-Civita method is indicated for application to nonzero cavitation numbers.

I. Introduction

WHEN a thin lifting hydrofoil moves in a liquid at a sufficiently large velocity or under a sufficiently extreme lifting condition, the pressures along the suction side of the profile will reach such low values that the liquid adjacent to the surface will vaporize, creating a cavity or bubble downstream of the hydrofoil. Under this condition, the streamlines bounding the cavitation region will originate at the leading and trailing edges of the hydrofoil, and the hydrofoil is said to be in a fully cavitated condition. The pressure in the cavity is approximately constant at the vapor pressure of the surrounding liquid. When the cavity pressure is less than the freestream pressure (finite cavitation number), the extent of the cavity is finite, and a highly unsteady wake flow develops downstream of the cavity similar to the wake behind a bluff body in a gaseous flow. As the cavity pressure is increased, the cavity extends further and further downstream and, in the absence of strong viscous effects, it will extend to infinity, attaining the steady flow configuration (shown in Fig. 1) as the cavity pressure approaches the freestream value (zero cavitation number).

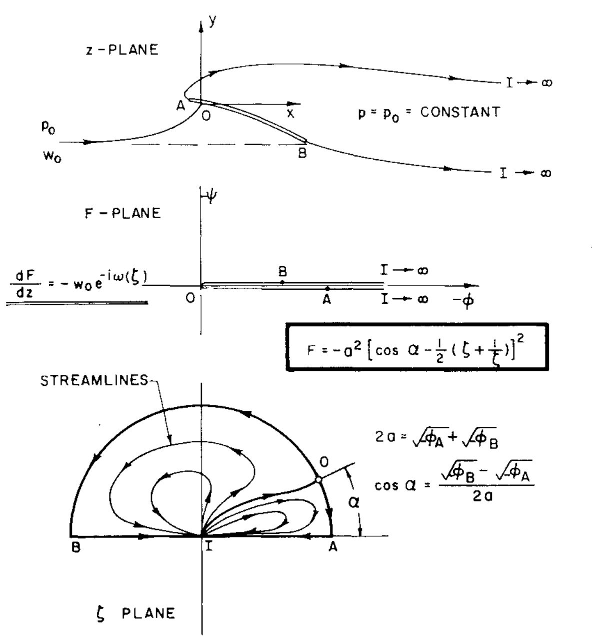

Fig. 1 Mapping for zero cavitation number: method of Levi-Civita.

In the present paper, we shall be concerned with the determination of optimum profile shapes for thin planar hydrofoils under fully cavitated conditions that have either minimum drag for a given lift or maximum lift for a given drag. The flow here will be assumed inviscid, and gravitational forces will be neglected. There have been previous investigations of this problem, using a linearized theory with planar boundary conditions (see, e.g., Tulin[1] and Parkin[2]). It is the purpose of the present paper to find an exact solution for the limiting case of zero cavitation number. This limiting case has considerable appeal as a starting point since the cavity configuration is extremely simple, and a conformal mapping procedure due to Levi-Civita (see Milne Thomson[3]) exists which is ideally suited for the optimization process. A numerical example is carried out to illustrate the procedure. Finally, modifications of the Levi-Civita method are indicated to extend the method to nonzero cavitation numbers.

II. Zero Cavitation Number: Method of Levi-Civita

A direct consideration of the boundary value problem in the physical plane is too complex. The governing flow differential equation, to be sure, is linear, being the Laplace equation for the complex potential function $F = \varphi + i\psi$, φ the velocity potential, and ψ the stream function; but the boundary conditions along the a priori unknown cavitation streamlines are nonlinear. The method of Levi-Civita avoids the difficulties in the physical plane by means of conformal mapping techniques. The starting point is the complex potential plane $F = \varphi + i\psi$ shown in Fig. 1. Here of course we do not know the relative locations of points A and B. The complex potential plane next is mapped to a ζ plane ($\xi = \xi + i\eta$) such that the flow domain is mapped into a semicircular sector of unit radius in the ζ plane. The necessary transformation is shown in the framed box in

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Fig. 1. The correspondence between the various planes is indicated by the same alphabetical letter. The task remaining is to find the transformation between the physical plane and the complex potential plane, or equivalently the ζ plane. This transformation next is expressed in terms of a function $\omega(\zeta)$ defined by

$$w_0 e^{-i\omega(\zeta)} = -(dF/dz) = we^{-i\theta} \tag{1}$$

or

$$\omega(\zeta) = \theta + i \log(w/w_0) \tag{2}$$

where w_0 is the cavitation velocity corresponding to the cavitation pressure, w the absolute value, and θ the flow inclination relative to the freestream direction of the local velocity vector. The essential problem of the Levi-Civita method is the determination of $\omega(\zeta)$, since, once it is known, the velocity vector field in the physical plane can be obtained from (1) by quadrature.

We seek now the most general form of $\omega(\zeta)$ which will result in a physically reasonable profile. The necessary criteria for defining $\omega(\zeta)$ are the following.

1) Along the cavitation boundaries AI and BI the velocity is constant and equal to w_0. Thus from Eq. (2), $\omega(\zeta)$ must be real along the real axis. Moreover, at the freestream point $\zeta = 0$, we have $\theta = 0$, so that $\omega(0) = 0$.

2) From (2) at the stagnation point $\omega(\zeta)$ must have a logarithmic singularity, and, as one passes through the stagnation point along the map of the wetted surface, the flow inclination θ must jump by π.

3) Lastly, within the semicircle including the boundary points but excluding the stagnation point, $\omega(\zeta)$ otherwise must be holomorphic.

The appropriate form of $\omega(\zeta)$ is

$$\omega(\zeta) = i \log \frac{\zeta - e^{i\alpha}}{1 - \zeta e^{i\alpha}} + \sum_{n=0}^{\infty} a_n \zeta^n \tag{3}$$

where $a_0 = \alpha + \pi$ from the requirement $\omega(0) = 0$ with α defined in Fig. 1, and $a_n (n \geq 1)$ are arbitrary real constants.† Each set of values for a_n will generate a different smooth profile in the physical plane.

The transformation between the ζ plane and the physical plane is obtained next by inserting (3) into (1) and carrying out the integration. Specifically for the wetted profile, we have $\zeta = e^{i\chi}$ $(0 \leq \chi \leq \pi)$, and we obtain

$$\begin{aligned} \frac{w_0}{a^2}\, x &= -2 \int_0^{\chi} e^{-\tau} \cos\theta(\cos\bar{x} - \cos\alpha) \sin\bar{x}\, d\bar{x} \\ \frac{w_0}{a^2}\, y &= -2 \int_0^{\chi} e^{-\tau} \sin\theta(\cos\bar{x} - \cos\alpha) \sin\bar{x}\, d\bar{x} \end{aligned} \tag{4}$$

where $\tau(\chi) = \log(w/w_0) = Im[\omega(\chi)]$, $\theta = Rl[\omega(\chi)]$, and a is defined in Fig. 1. Here the origin of the physical plane has been taken at the leading edge.

The coordinates of the cavitation boundaries next are given by

$$\begin{aligned} \frac{w_0}{a^2}(x - x_{\pm}) &= \frac{1}{2} \int_{\pm 1}^{\xi} \cos\theta\left(\zeta + \frac{1}{\zeta} - 2\cos\alpha\right) \times \left(\zeta - \frac{1}{\zeta}\right) \frac{d\zeta}{\zeta} \\ \frac{w_0}{a^2}(y - y_{\pm}) &= \frac{1}{2} \int_{\pm 1}^{\xi} \sin\theta\left(\zeta + \frac{1}{\zeta} - 2\cos\alpha\right) \times \left(\zeta - \frac{1}{\zeta}\right) \frac{d\zeta}{\zeta} \end{aligned} \tag{5}$$

Here $(x_{\pm}, y_{\pm})$ is the coordinate of the profile extremity located at $\xi = \pm 1$.

Finally the drag and lift may be calculated using the Blasius formula

$$D + iL = \frac{i}{2} \rho_0 w_0 a^2 \oint_c e^{i\omega(\zeta)} \frac{dw}{d\zeta} d\zeta$$

where ρ_0 is the flow density, contour C is the unit circle, and $\omega(\zeta)$ is to be continued analytically into the lower half circle by using the reflection principle of Schwartz. The integrand has a simple pole at the origin, and, by the residue method, the foregoing integral can be evaluated with the result

$$\tilde{D} = \frac{D}{(\pi/4)\rho_0 w_0 a^2} = [\omega'(0)]^2$$

$$\tilde{L} = \frac{L}{(\pi/4)\rho_0 w_0 a^2} = [4\omega'(0) \cos\alpha - \omega''(0)]$$

where the primes denote differentiations; or, using (3), we obtain

$$\begin{aligned} \tilde{D} &= (a_1 - 2 \sin\alpha)^2 \\ \tilde{L} &= 2(-\sin 2\alpha + 2a_1 \cos\alpha - a_2) \end{aligned} \tag{6}$$

Of the infinite set of coefficients $\{a_n\}$ it thus is seen that the drag depends only upon the coefficient a_1, whereas the lift depends only upon a_1 and a_2.

III. Optimization

The introduction of an explicit form of $\omega(\zeta)$, as given by (3), immediately reduces the optimization problem from a variational problem to one of ordinary calculus. From (5) it is immediately clear that the problem of maximizing the lift for a given drag does not possess a solution since the lift depends only linearly upon the coefficients a_1 and a_2.

For the problem of finding the profile of minimum drag and possessing a prescribed lift, the method of Lagrangian multipliers is the logical starting point. However, an inspection of (6) shows immediately that a minimum drag profile of zero drag can be found by choosing $a_1 = 2 \sin\alpha$; the desired lift then can be obtained by choosing the appropriate combination of values of a_2 and α. Since the forces depend only upon the coefficients a_1 and a_2 for a given value of α, it is clear that the optimum profile is not unique, and a whole class of optimum profiles may be found, each characterized by a different set of coefficients $\{a_n\}$ where $n \geq 2$.

For the simplest illustrative example we shall take all coefficients $a_n = 0$ except a_0 and a_1 and then select the value of α to obtain the desired lift. Figures 2 and 3 show the resulting profiles, pressure distributions, and the cavitation streamlines for the three cases where α is $\pi/16$, $\pi/8$, and $\pi/4$. These curves correspond to $\tilde{L} = 0.766$, $2^{1/2}$, and 2, respectively, for $\alpha = \pi/16$, $\pi/8$, and $\pi/4$; or, if we take the reference length as the length of the chord joining the profile extremities, they correspond to the lift coefficients $C_L = 0.355$, 0.850, and 2.250.

In these examples, wetted surface pressures arise which are less than the cavity pressure. By definition we cannot allow pressures lower than the cavitation pressure even though the freestream pressure is an unrealistically high cavitation pressure. On this basis the foregoing flows must be disqualified as permissible flows. On the other hand, one can raise the cavity pressure by bleeding sufficient exhaust gas into the cavity to attain the freestream pressure. In this case the previous zero drag profile will be of interest provided, of course, the wetted pressures do not drop below a realistic vapor pressure and the volume of bleed gas required is not excessive.

† The coefficients a_n of course must be such as to insure proper convergence of the series.

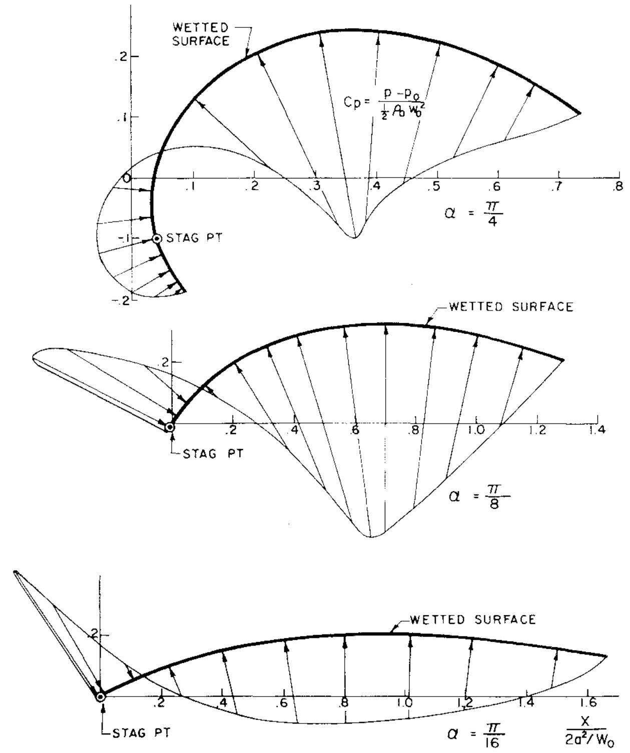

Fig. 2 Optimum profiles and pressure distributions.

It may be possible, however, that by superimposing other terms in the series in (3) for $\omega(\zeta)$, the wetted pressures might be constrained never to be less than the cavity pressure. To explore this possibility, let us first write the expression for the velocities along the wetted surface. From (2) and (3) we obtain

$$\log(w/w_0) = \tfrac{1}{2}\log\{[1 - \cos(\chi - \alpha)/[1 - \cos(\chi + \alpha)]\} + 2\sin\alpha\sin\chi + a_2\sin 2\chi + a_3\sin 3\chi + \ldots \quad (7)$$

In (7) we now multiply both sides of the equation by $\sin\chi$ and integrate the resulting expression from 0 to π. We obtain

$$\int_0^\pi \log\left(\frac{w}{w_0}\right)\sin\chi d\chi = \pi\sin\alpha + \frac{1}{2}\int_0^\pi \log \left[\frac{1 - \cos(\chi - \alpha)}{1 - \cos(\chi + \alpha)}\right] \times \sin\chi d\chi \quad (8)$$

The integral on the right side may be evaluated with the resulting value of $-2\pi\sin\alpha$. That is, (8) now becomes

$$\int_0^\pi \log\left(\frac{w}{w_0}\right)\sin\chi d\chi \equiv 0$$

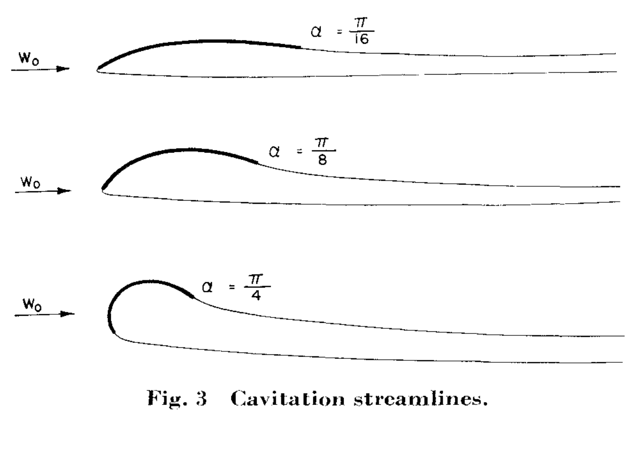

Fig. 3 Cavitation streamlines.

This means that either $w/w_0 = 1$ along the entire wetted surface, which is, of course, impossible because of the requirement of a stagnation point, or that w/w_0 must be both greater or smaller than 1 since $\sin\chi \geq 0$ for $0 \leq \chi \leq \pi$; that is, we must require pressures less than the cavitation pressure along portions of the wetted surface. Thus it is seen that zero drag profiles do not exist among the class of hydrofoils with regular profiles, as given by (3), which have wetted pressures everywhere not less than the cavity pressure.

IV. Nonzero Cavitation Numbers

As briefly discussed in the Introduction, the fully cavitated flow at finite cavitation numbers is considerably more complex than the case for zero cavitation number. Nevertheless, there have been numerous inviscid steady models proposed for this case. Representative examples of the various models are given by Riabouchinsky's image model, Prandtl's re-entrant jet model, and Roshko's dissipation model. These are described in detail by Wu,[4] who has found that these models essentially give the same flow behavior in the vicinity of the profile, differing significantly only far downstream.

In the present discussion we shall adopt the dissipation model of Roshko which has been modified more recently by Wu.[5] It may be recalled that the flow for a finite cavitation number was composed of a finite, relatively steady cavity region at essentially constant pressure followed by a highly unsteady viscous wake further downstream. The model of Roshko essentially replaces the above flow by an inviscid, steady one in which the cavity and the viscous wake regions are separated from the external, essentially inviscid flow by two steady separation streamlines, which originate at the leading and trailing edges of the hydrofoil. The pressure along the initial portions of the separation streamline near the hydrofoil is assumed to be constant at the cavitation pressure, whereas further downstream it is assumed to increase continuously to the freestream value far downstream.

Using the foregoing model, let us now revise the method of Levi-Civita for nonzero cavitation numbers. The starting point is the function $\omega(\zeta)$. The form given by (3) now must be modified so that the velocity along the cavitation streamlines AI and BI, instead of being constant at the value w_0, is allowed to vary from the cavitation velocity w_0 at $\zeta = \pm 1$ to the freestream velocity w_∞ at $\zeta = 0$. Consider the modified form of $\omega(\zeta)$

$$\omega(\zeta) = i\log\frac{\zeta - e^{i\alpha}}{1 - \zeta e^{i\alpha}} + \sum_{n=0}^{\infty} a_n\zeta^n + iQ(\zeta) \quad (9)$$

where we have added the additional term $Q(\zeta)$, which is a regular function of ζ which is real for real values of ζ; the a_n's again are real constants.

Along the separation streamline (real values of ζ), the velocities now are given by

$$\log(w/w_0) = Q(\zeta)$$

Thus $Q(\zeta)$ must be a continuous function varying from zero at $\zeta = \pm 1$ to $\log(w_\infty/w_0)$ at $\zeta = 0$. A guide to a satisfactory form of $Q(\zeta)$ may be obtained by considering a simple case as a flat plate that has been calculated, for example, by Wu.[5]

Let us now examine how the addition of $iQ(\zeta)$ to $\omega(\zeta)$ will affect the remaining considerations. Since $Q(\zeta)$ is a regular function, it will not affect the nature of the stagnation point. At the freestream point $\zeta = 0$, we will have the correct condition if we again select $a_0 = \alpha + \pi$.

The new expressions for lift and drag with the addition of $iQ(\zeta)$ now become

$$\tilde{D} = [\omega_0'(0)]^2 - [Q'(0)]^2 - 4\cos\alpha Q'(0) + Q''(0)$$

$$\tilde{L} = 4\cos\alpha\omega_0'(0) - \omega_0''(0) + 2\omega_0'(0)Q'(0)$$

Here ω_0 denotes the expression for $\omega(\zeta)$ for the case of zero cavitation number.

The previous expressions of course are more complex than those for zero cavitation number, but they may be simplified without difficulty by selecting a $Q(\zeta)$ for which either or both $Q'(0)$ and $Q''(0)$ are equal to zero. The further procedure for finding the minimum drag profile would follow as for the case of zero cavitation number.

V. Conclusions

Optimal profiles for zero cavitation number have been sought among the class of profiles having an analytic shape, as given by (3), which have either minimum drag for a given lift or maximum lift for a given drag. The results indicated that optimal profiles that have maximum lift for a given drag do not exist. On the other hand, profiles that have minimum drag of zero for a given lift have been found, but they were not unique, and possessed wetted pressures along portions of the profile which were less than the cavity pressure. Further investigation showed that it was impossible to find optimum profiles with wetted pressures everywhere not less than the cavity pressure. This result is in agreement with that obtained for the linearized case by Tulin[1] and Parkin,[2] who also found that the optimum profile shape could not be regular.

The situation for nonzero cavitation numbers is unclear at the moment and must await the development of a model for these flows based upon a modification of the Levi-Civita procedure outlined in the last section. There are, of course, other existing models for these flows, but they appear to be less suitable for the optimization procedure.

References

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